COMMENTS ON THE PAPER BY A. FERRI, "RECENT THEORETICAL WORK IN SUPERSONIC AERODYNAMICS AT THE POLYTECHNIC INSTITUTE OF BROOKLYN"

(ZAMECHANIE K DOKLADY A. FERRI "NOVAIA TEORETICHESKAIA RABOTA PO SVERKHZVYKOVOI AÈRODINAMIKE V BRUKLINSKOM POLITEKHNICHESKOM INSTITUTE")

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A large part of the paper by Ferri [1] is devoted to a discussion of supersonic conical flows, for high Mach numbers and for bodies which produce large flow disturbances. In particular, much space is devoted to the discussion of the properties of conical flows in the vicinity of the parabolic ("sonic") line. In the present note it is shown that the behavior of a conical flow adjoining a uniform flow along a Mach cone (parabolic line) cannot be described by means of the expansions proposed by Ferri and his co-workers. This circumstance eliminates the basis of the conclusion about the existence of intermediate, transonic zones adjoining the Mach cone, as obtained by Ferri in his consideration of supersonic flow over triangular conical wings with cross-sections of rhombic type. The question of the possibility of the appearance of such regions is discussed.

We shall consider for definiteness a triangular conical wing having a cross-section of rhombic shape, in a uniform supersonic flow at zero angle of attack (Fig. 1). Due to the symmetry of the flow, we shall consider only the region $x < 0$, $y > 0$. We assume that the leading edges of the wing are supersonic. Since the resulting flows will be conical, the velocity components and the entropy depend only on $\xi = x/z$, $\eta = y/z$, or, if we use a spherical system of coordinates with origin at the point O (Fig. 1), only on angular variables.

The picture of the flow over the wing in the $\xi\eta$-plane is shown in Fig. 2. At the leading edge there originates a plane shock wave, represented on Fig. 2 by the segment *4-3*; behind it comes a region of uniform flow adjacent to the wing surface (the line *4-5-1*). The uniform flow is separated from the conical flow by the corresponding Mach cone (the

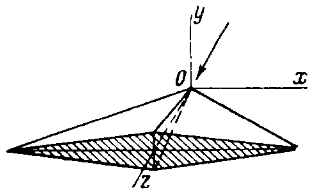

Fig. 1.

elliptic arc *3-5*). The curve *3-2* corresponds to a curved shock, originating in the central part of the wing.

The system of equations which describes conical irrotational (rotational) flows is of hyperbolic type, if the projection of the velocity on the plane perpendicular to the radius vector of a point in the xyz-space is greater than the local speed of sound; otherwise it is elliptic (mixed: two imaginary characteristics and two real ones).

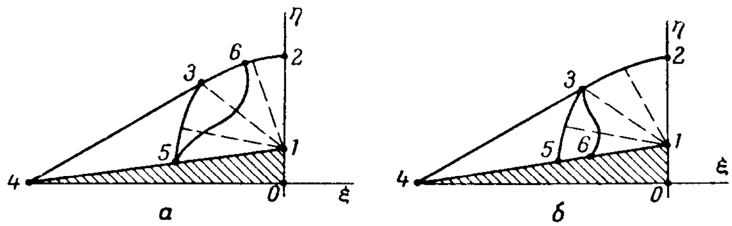

Fig. 2.

In the uniform flow region *4-5-3* (Fig. 2) the solution is of hyperbolic type. The arc of the Mach cone *3-5* is simultaneously a parabolic line and a characteristic curve. In Ferri's paper it is maintained that adjacent to the uniform flow, along the arc of the Mach cone *3-5*, there is a region with a hyperbolic solution (transonic region) *3-5-6* (Fig. 2). This region joins, along another parabolic line *5-6* (in Fig. 2*a*) *3-6* (in Fig. 2*б*), a region where the solution is of elliptic type (*1-2-6-5* in Fig. 2*a* and *1-2-3-6* in Fig. 2*б*). The transonic regions may be of two types (Figs. 2*a* and 2*б*). The behavior of the solution in these regions is described by expansions which will be discussed below.

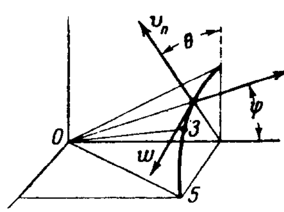

Fig. 3.

Following Ferri, we introduce a spherical coordinate system ψ, θ, r, with axis in the direction of the uniform flow in the region *4-5-3* (Figs. 2 and 3).

The Mach cone *3-5* (Fig. 2) maps on to the circumference $\psi = \psi^*$ of the unit sphere, where $\sin \psi^* = M^{-1}$ and M is the Mach number of the uniform flow in region *4-5-3* (Fig. 2).

Instead of v_n, w we introduce new variables $u = \sqrt{v_n^2 + w^2}$ and β, where $\operatorname{tg} \beta = v_n/w$; and we distinguish variables on the Mach cone by a star, thus

$$v_n^* = -a^*, \qquad \beta^* = -\frac{1}{2}\pi$$

$$v_r^* = a^* \sqrt{M^2 - 1}, \qquad w^* = 0$$

(a^*) is the speed of sound in the uniform flow). In Ferri's paper an expansion is obtained for irrotational flow in the neighborhood of an arbitrary elliptic (sonic) line; for the case of the arc of the Mach curve in the chosen system of coordinates it is written in the form

$$u = a^* \left(1 + K_{12} A_{12} + m_{12}^2 A + \ldots\right) \tag{1}$$

$$v_r = a^* \sqrt{M^2 - 1} \left\{1 + \frac{2}{3}(M^2 - 1)^{-1/2} (\gamma + 1)^{1/2} K_{12}{}^{1/2} A_{12}{}^{3/2} + \frac{\gamma - 1}{6} A_{12}^2 + \right.$$

$$+ \frac{1}{5}(M^2 - 1)^{-1/2} \left(\frac{\gamma + 1}{K_{12}}\right)^{1/2} \left[\frac{(\gamma - 1)^2}{18(\gamma + 1)}(M^2 - 1) + \frac{K_{12}^2}{2} + m_{12}\right] A_{12}^{5/2} +$$

$$\left. + \left[\frac{(\gamma - 1)^3}{108 K_{12}}(M^2 - 1)^{1/2} \left(\frac{1}{5(\gamma + 1)} - (M^2 - 1)^{1/2}\right) + \frac{\gamma - 1}{45 K_{12}} \left(\frac{53 - 5\gamma}{4} K_{12}{}^2 - m_{1_2}\right)\right] A_{12}^3 + \ldots\right.$$

$$\beta = -\frac{\pi}{2} \pm \frac{2}{3}(\gamma + 1)^{1/2} K_{12}^{5/2} A_{12}^{3/2} \pm K_{12}(M^2 - 1)^{1/2} \frac{4\gamma + 5}{6} A_{12}^2 + \ldots$$

$$\psi = \psi^* - \frac{2}{3}(\gamma + 1)^{1/2} K_{12}^{1/2} A_{12}{}^{3/2} - \frac{\gamma - 1}{6}(M^2 - 1)^{1/2} A_{12}^2 + \ldots$$

$$\theta = \theta^* \mp M A_{12} \mp \frac{\gamma + 1}{4} M K_{12} A_{12}^2 + \ldots$$

Here A_1, A_2 are curvilinear coordinates along characteristics defined by the equations

$$\frac{d\psi}{dA_{12}} = \sin(\beta \pm \mu), \qquad \frac{\sin\psi\, d\theta}{dA_{12}} = \cos(\beta \pm \mu), \qquad \sin\mu = \frac{a}{u}$$

and originating on the parabolic line; the index 1 and superscript in the formulas refer to the first family of characteristics, and the index 2 and subscript refer to the second family; K_{12}, m_{12} are coefficients of the series expansion of u in powers of A_{12} along the characteristics; ψ^*, θ^* are the values of ψ and θ for a fixed characteristic on the parabolic line; γ is the ratio of specific heats. The dots denote higher order terms.

From equation (1) it is evident that K cannot become zero, since otherwise one of the coefficients of $A_{12}{}^{5/2}$ or $A_{12}{}^3$ in the expansion for

v_r would become infinite. It also follows from equation (1) that along characteristics

$$A_{12} = \left(\frac{3}{2}\right)^{2/3} (\gamma + 1)^{-1/3} K_{12}^{-1/3} (\psi^* - \psi)^{2/3} + \ldots \tag{2}$$

Substituting (2) in (1) we obtain

$$u = a^* \left[1 + \left(\frac{3}{2}\right)^{2/3} (\gamma + 1)^{-1/3} K_{12}^{2/3} (\psi^* - \psi)^{2/3} + \ldots\right] \tag{3}$$

Now we shall consider not one characteristic but the whole family, e.g. the first, then K_1 is a function of ψ and θ, $K_1 = K_1(\psi, \theta)$, which remains finite as $\psi \to \psi^*$ in arbitrary manner, since the family of characteristics fills the neighborhood of the parabolic line (on one side). We write

$$\lim_{\psi \to \psi^*} K_1(\psi, \theta) = K(\theta) \neq 0$$

Then

$$K_1(\psi, \theta) = K(\theta) - [K(\theta) - K_1(\psi, \theta)] = K(\theta) + \ldots$$

and (3) may be written in the form

$$u = a^* \left[1 + \left(\frac{3}{2}\right)^{2/3} (\gamma + 1)^{-1/3} K(\theta)^{2/3} (\psi^* - \psi)^{2/3} + \ldots\right] \tag{4}$$

which is valid for arbitrary ψ and θ in the vicinity of $\psi = \psi^*$.

Analogously we obtain from (1) and (2)

$$\beta = -\frac{1}{2}\pi + \left(\frac{2}{3}\right)^{1/3} (\gamma + 1)^{1/3} K^2(\theta)(\psi^* - \psi) + \ldots \tag{5}$$

$$v_r = a^* \sqrt{M^2 - 1} + a^* (\psi^* - \psi^*) \tag{6}$$

The equation of continuity, after transformation with the help of the equations of momentum and energy, is written in the form

$$\begin{gathered} v_r \left(2 - \frac{v_n^2 + w^2}{a^2}\right) + v_n \operatorname{ctg}\psi + \left(1 - \frac{v_n^2}{a^2}\right)\frac{\partial v_n}{\partial \psi} + \\ + \left(1 - \frac{w^2}{a^2}\right)\frac{\partial w}{\sin\psi\, \partial\theta} - \frac{v_n w}{a^2}\left(\frac{\partial w}{\partial \psi} + \frac{\partial v_n}{\sin\psi\, \partial\theta}\right) = 0 \end{gathered} \tag{7}$$

For irrotational flow

$$\frac{\partial v_n}{\partial \theta} = \frac{\partial (w \sin\psi)}{\partial \psi} \tag{8}$$

We put (8) in (7), multiply the result by $\sin\psi(1 - w^2/a^2)^{-1}$, integrate with respect to θ from θ_0 to θ, make the substitution $v_n = u \sin\beta$, $w = u\cos\beta$, (the purpose of the integration is to eliminate the derivative $\partial w/\partial\theta$) and obtain the result

$$\int_{\theta_0}^{\theta} \sin\psi\,(u^2\cos^2\beta - a^2)\Big\{v_r\,(2a^2 - u^2) + u\sin\beta\,(a^2 - u^2\cos^2\beta)\operatorname{ctg}\psi +$$

$$+ [(a^2 - u^2\sin^2\beta)\sin\beta - 2u^2\sin\beta\cos^2\beta]\frac{\partial u}{\partial\psi} + [(a^2 - u^2\sin^2\beta)\cos\beta +$$

$$+ 2u^2\sin^2\beta\cos\beta]\,u\,\frac{\partial\beta}{\partial\psi}\Big\}\,d\theta + u\,(\psi,\theta_0)\cos\beta\,(\psi,\theta_0) - u\,(\psi,\theta)\cos\beta\,(\psi,\theta) = 0 \qquad (9)$$

Using equations (4), (5), (6) and their derivatives with respect to ψ (terms indicated by dots do not affect the result) it is easy to prove by direct substitution that, in the vicinity of $\psi = \psi^*$, equation (9) has the form

$$\frac{a^*}{M}\left(\frac{3}{2}\right)^{1/3}(\gamma+1)^{1/3}\int_{\theta_0}^{\theta} K^{4/3}(\theta)\,d\theta\,(\psi^* - \psi)^{1/3} + \ldots = 0 \qquad (10)$$

Dividing equation (10) by $(\psi^* - \psi)^{1/3}$ and letting ψ approach ψ^*, we find that

$$\int_{\theta_0}^{\theta} K^{4/3}(\theta)\,d\theta \equiv 0 \qquad (11)$$

Differentiating equation (11) with respect to θ, assuming piecewise continuity of $K(\theta)$, we obtain

$$K(\theta) \equiv 0 \qquad (12)$$

which contradicts the condition obtained earlier, that $K \neq 0$.

This contradiction shows that the expansion (1) which is valid in the vicinity of an isolated parabolic ("sonic") point of the type $\beta = -\pi/2$, does not represent a solution in the case where these points continuously fill the whole curve (Mach cone).

The author of the present note investigated the behavior of conical flows in the vicinity of the Mach cone, with natural assumptions regarding the smoothness of the solution [2], and found that the expansion he obtains for F leads to an elliptic type of solution in the vicinity of the Mach cone. Other types of solution were not found, and probably do not exist.

As can be understood from Ferri's paper, the second parabolic line *3-6* (Fig. 2*b*) was located in the following way. (Let us note at once that we do not consider the case shown in Fig. 2*a*, since we do not know any solutions of hyperbolic type in the vicinity of the Mach cone.) The form of the curvilinear shock *3-2* (Fig. 2*b*) was given; this made it possible, from the shock conditions and the equations of conical flow, to find the values of the velocities and their normal derivatives on the line of the shock, and thus calculate the values of the velocities at points near the shock.

From these values of the velocities the equations of conical flow were again used to find the normal derivatives, from which the values of the velocities were found at the next points, and so on. This leads to the appearance of the second parabolic line.

Here it is necessary to draw attention to an important point, that along the line of constant entropy *3-1* (Fig. 2*b*) there is a joining of irrotational and rotational conical flows. (In what follows we shall call lines of constant entropy streamlines; they are indicated by dotted lines in Fig. 2). We now investigate the properties of the transition across such a line. The system of equations describing rotational, conical flows with constant (total) enthalpy, has the form:

$$
\begin{aligned}
&L_1 = (u-\xi w)\left(\frac{u^2+v^2+w^2}{2}\right)_\xi + (v-\eta w)\left(\frac{u^2+v^2+w^2}{2}\right)_\eta + a^2(\xi w_\xi + \eta w_\eta - u_\xi - v_\eta) = 0\\
&L_2 = w(\xi u_\xi + \eta v_\xi + w_\xi) + (v-\eta w)(v_\xi - u_\eta) + a^2 s_\xi = 0\\
&L_3 = w(\xi u_\eta + \eta v_\eta + w_\eta) + (u-\xi w)(u_\eta - v_\xi) + a^2 s_\eta = 0 \qquad\qquad (13)\\
&L_4 = (u-\xi w)s_\xi + (v-\eta w)s_\eta = 0
\end{aligned}
$$

Here $s = S[(\gamma-1)\gamma c_v]^{-1}$, S is the entropy, c_v is the specific heat at constant volume, u, v, w are velocity components in a cartesian coordinate system, a is the speed of sound,

$$a^2 = a_0^2 - {}^1/_2(\gamma-1)(W_0^2 - u^2 - v^2 - w^2), \qquad \xi = x/z, \qquad \eta = y/z$$

where a_0, W_0 are the speed of sound and the flow speed at a certain point.

Equations (13) are a combination of the equations of momentum, continuity and energy.

Let us substitute for $L_2 = L_3 = 0$ the following equations which are more convenient for what follows:

$$
\begin{aligned}
L_5 = L_2(u-\xi w) + L_3(v-\eta w) = \xi[(u-\xi w)u_\xi + (v-\eta w)u_\eta] +\\
+ \eta[(u-\xi w)v_\xi + (v-\eta w)v_\eta] + (u-\xi w)w_\xi + (v-\eta w)w_\eta = 0
\end{aligned}
$$

$$
\begin{aligned}
L_6 = -L_2(v-\eta w) + L_3(u-\xi w) = w\{\xi[(u-\xi w)u_\eta - (v-\eta w)u_\xi] +\\
+ \eta[(u-\xi w)v_\eta - (v-\eta w)v_\xi] + (u-\xi w)w_\eta - (v-\eta w)w_\xi\} +\\
+ [(u-\xi w)^2 + (v-\eta w)^2](u_\eta - v_\xi) + a^2[(u-\xi w)s_\eta - (v-\eta w)s_\xi] = 0
\end{aligned}
$$

The system (13) is equivalent to system (14):

$$L_1 = L_4 = L_5 = L_6 = 0 \qquad (14)$$

From the form of the equations $L_4 = 0$, $L_5 = 0$ it follows at once that the streamlines, defined by

$$\frac{d\xi}{u-\xi w}=\frac{d\eta}{v-\eta w}$$

are double characteristics of the system (14). The remaining two characteristics, as may be shown, are determined by the same equations as in the case of irrotational flow

$$\frac{d\eta}{d\xi}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

$$A=a^2(1+\xi^2)-(u-\xi w)^2,\ B=a^2\xi\eta-(u-\xi w)(v-\eta w),\ C=a^2(1+\eta^2)-(v-\eta w)^2$$

The characteristics do not coincide with streamlines.

Since the streamlines are characteristics of (14), derivatives of s, as well as of u, v, w, may have discontinuities across them. We introduce the variables ρ, σ by means of the formulas $\xi = \xi(\rho, \sigma)$, $\eta = \eta(\rho, \sigma)$, such that ρ = const along a streamline and σ = const along a normal trajectory to it.

In the new variables, system (14) is written as follows:

$$L_1=[(v-\eta w)\xi_\rho-(u-\xi w)\eta_\rho]\left(\frac{u^2+v^2+w^2}{2}\right)_\sigma+a^2[\xi(\eta_\sigma w_\rho-\eta_\rho w_\sigma)+$$
$$+\eta(\xi_\rho w_\sigma-\xi_\sigma w_\rho)+\eta_\rho u_\sigma-\eta_\sigma u_\rho+\xi_\sigma v_\rho-\xi_\rho v_\sigma]=0$$

$$L_4=s_\sigma=0,\qquad L_5=\xi u_\sigma+\eta v_\sigma+w_\sigma=0 \tag{15}$$

$$L_6=w[(v-\eta w)\eta_\sigma+(u-\xi w)\xi_\sigma](\xi u_\rho+\eta v_\rho+w_\rho)+[(u-\xi w)^2+$$
$$+(v-\eta w)^2](\xi_\sigma u_\rho-\xi_\rho u_\sigma+\eta_\sigma v_\rho-\eta_\rho v_\sigma)+a^2[(v-\eta w)\eta_\sigma+(u-\xi w)\xi_\sigma)]s_\rho=0$$

From system (15) it is clear that specification of u, v, w as a function of $\sigma(s = s_0)$ along a streamline $\rho = \rho_0$, does not uniquely determine the normal derivatives u_ρ, v_ρ, w_ρ, s_ρ, that is, they may have a discontinuity here.

Inasmuch as the streamline *3-1* (Fig. 2*b*) separates an irrotational flow (s = const) and a rotational one ($s = s_\rho$), therefore s_ρ (or higher derivatives of s with respect to ρ) have a discontinuity, and therefore the derivatives u_ρ, v_ρ, w_ρ (or their higher order derivatives with respect to ρ) must have a discontinuity, as may be seen from the equation $L_6 = 0$ in system (15).

For irrotational flow $\quad \xi u_\rho+\eta v_\rho+w_\rho=0$

$$\xi_\sigma u_\rho-\xi_\rho u_\sigma+\eta_\sigma v_\rho-\eta_\rho v_\sigma=(\xi u_\rho+\eta v_\rho+w_\rho)_\sigma-(\xi u_\sigma+\eta v_\sigma+w_\sigma)_\rho=0$$

Evidently, a discontinuity in the derivatives of u, v, w was not accounted for in going across the streamline *3-1*, and the smooth continuation of the solution from the shock across the line *3-1* always led to the appearance of the second parabolic ("sonic") line *3-6* (Fig. 2*b*).

From the above, it follows that the question of the appearance of a second sonic line, and thus, an intermediate transonic region, requires further investigation, and it is possible that they do not exist in general, in an exact solution. (Possibly at point *3*, Fig. 2, a forked shock is produced.)

We note in passing, that in the article by Fowell [3], in the numerical solution of the problem of a triangular plane wing, the possibility of a discontinuity in the velocity derivatives on a line analogous to *3-1* was not taken into account. For that reason it is not clear to what extent the calculations of Fowell take into account the flow vorticity, and whether they may be considered an "exact" solution.

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